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Attitude Motion of Asymmetric Dual-Spin Spacecraft

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An exact analytical solution is obtained for the attitude motion of a class of asymmetric dual-spin spacecraft which may be modeled as free systems of two rigid bodies, one asymmetric and one axisymmetric. The type system considered constitutes a gyrostat. The symmetry axis of the axisymmetric rigid body and either the axis of the largest, or that of the smallest, moment of inertia of the system are assumed to be parallel. Zero internal torque is assumed also. The integral of rotational angular momentum of the system, an integral of energy of the system and the constancy of the symmetry-axis component of rotational angular momentum of the symmetric body are used to reduce the solution of the problem to quadratures, which are evaluated in terms of Jacobian elliptic functions and elliptic integrals. Internal resonances, which may exist in imperfect spacecraft, are discussed. Resonance curves, constructed using the exact solution, are presented for a typical, asymmetric, dual-spin spacecraft.

Introduction

THE analytical solution of problems of rigid-body attitude dynamics is of academic and, often, practical interest. The specific problem of determining the attitude of a system of two rigid bodies which are coupled so that there is one degree of freedom in relative rotation, and the inertia characteristics of the system are invariant is one that attracts both types of interest, since dual-spin spacecraft can be modeled, at least as a first approximation, by such "gyrostats."

An analytical solution to the problem of determining the attitude of a free axisymmetric gyrostat with a free rotor (see, for example, Ref. 1, pp. 218-220) is relatively straightforward. Furthermore, Kane[2] has given part of the solution for the attitude motion of a uniaxial gyrostat with a driven rotor. When one of the bodies is asymmetric, however, it appears that a complete analytical solution in closed form has not been presented in the open literature, although Leimanis (Ref. 1, pp. 221-232) has reduced the problem to quadratures. This problem is of practical interest because some dual-spin spacecraft have rotors and/or platforms that are asymmetric,[3-5] and asymmetries may result in problems with internal resonance when imperfections such as dynamic imbalances are present.[3-5]

Leimanis's solution is apparently based partially on that of Masaitis,[6] who obtained solutions only for the angular velocity components of the asymmetric body. Although Leimanis has added to Masaitis's efforts, the solution in Ref. 1 is still "incomplete" in the sense that two of the Eulerian angles, and the angle of relative rotation, are not obtained as explicit functions of time. Leimanis states, (Ref. 1, p. 323), in referring to this fact, that "[d]ue to the complexity of the integrals involved . . . the complete solution sketched above can be obtained only by numerical means." It is shown herein that analytical solutions can be found for the two Eulerian angles in question, and the angle of relative rotation of the rotor.

In this paper, a closed-form, analytical solution is obtained for the attitude motion of a free system of two rigid bodies, one asymmetric and one axisymmetric, which are coupled together in such a manner that the axisymmetic body may rotate freely with respect to the other about its axis of symmetry, which is parallel to the axis of one of the extreme moments of inertia of the asymmetric body. Three immediate integrals exist because of the constancy of 1) the rotational angular momentum of the system about its center of mass, 2) the kinetic energy of the system due to rotation about its center of mass, and 3) the component of rotational angular momentum of the axisymmetric body about its symmetry axis. These are used to reduce the solution of the problem to quadratures, one of which has been evaluated previously.[1,6] However, the quadratures for two of the three Eulerian angles, and the angle of relative rotation, are new. The solutions for the proper motion of the asymmetric body and the angle of relative rotation of the axisymmetric body can be used to determine conditions under which internal resonance exists.[3-5] Exact resonance curves for a typical spacecraft are presented and compared with a previously obtained approximate curve.[4]

Equations of Motion

The system considered is illustrated in Fig. 1. The bodies are denoted as R and S. Either may be the rotor or the platform; however, body S is the axisymmetric one. The dextral, orthogonal, coordinate systems $C_R a_1 a_2 a_3$ and $C_S b_1 b_2 b_3$ are centroidal principal systems of R and S, respectively. The symmetry axis of body S, b_1, is shown collinear with a_1. However, all results given herein are valid if b_1 and a_1 are only parallel. The angle of rotation of body S with respect to body R is α. The $Cx_1x_2x_3$ system has its origin at the system center of mass, and is at all times aligned with the $C_R a_1 a_2 a_3$ system.

All but one of the attitude equations of the system may be obtained from Eqs. (20) of Ref. 4 by deleting all terms due to dynamic imbalance. These are

$$\dot{H}_1 = [(B-C)/(BC)]H_2H_3 \tag{1a}$$

$$\dot{H}_2 = \{[(C-A)/(AC)]H_1 - P_\alpha/A\}H_3 \tag{1b}$$

$$\dot{H}_3 = \{[(A-B)/(BA)]H_1 + P_\alpha/A\}H_2 \tag{1c}$$

and

$$\dot{P}_\alpha = 0 \tag{1d}$$

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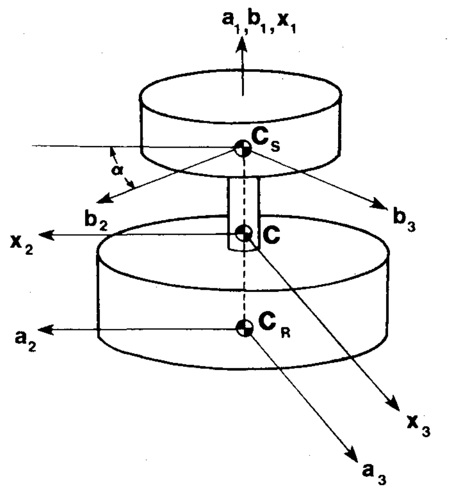

Fig. 1 Spacecraft model.

Reference 4 also provides the kinematic equation,

$$\dot{\alpha}=P_\alpha[1/A+1/B_1]-H_1/A \tag{1e}$$

Here, H_j is the x_j-component of the rotational angular momentum of the system about its center of mass, and P_α is the x_1-component of the rotational angular momentum of body S. In addition, the moments of the inertia A and B_1 are those of bodies R and S, respectively, about the x_1-axis. Finally, B and C are the moments of inertia of the system about the x_2- and x_3-axes, respectively.

In deference to readers more comfortable with angular velocity components as variables, and for later use also, it is noted that

$$H_1=A\omega_1+P_\alpha \qquad H_2=B\omega_2 \qquad H_3=C\omega_3 \tag{2}$$

where ω_j is the x_j-component of ω, the angular velocity of the $Cx_1x_2x_3$ system. Also,

$$P_\alpha=B_1(\Omega+\omega_1) \tag{3}$$

where $\Omega=\dot{\alpha}$.

Since there is no external torque, the rotational angular momentum of the system, H, is constant and the attitude of the $Cx_1x_2x_3$ system can be defined using the Eulerian angles Ψ, Θ, and Φ as shown in Fig. 2, where the nonrotating $CX_1X_2X_3$ system has its X_1-axis aligned with H. Furthermore, from geometry (see Fig. 2),

$$\cos\Theta=H_1/H \tag{4}$$

and

$$\tan\Phi=H_2/H_3 \tag{5}$$

so that, in principle, only an equation for Ψ (the precession angle), is needed in addition to Eqs. (1) [and, of course, Eqs. (4) and (5)] to completely define the system's attitude. From Poisson's kinematic equations, Eqs. (2) and the equations,

$$H_1=H\cos\Theta \qquad H_2=H\sin\Theta\sin\Phi \qquad H_3=H\sin\Theta\cos\Phi \tag{6}$$

which are, obviously, equivalent to Eqs. (4) and (5), it follows that

$$\dot{\Psi}=H(H_2^2/B+H_3^2/C)/(H^2-H_1^2) \tag{7}$$

Immediate Integrals

The fact that H is constant has already been mentioned. The constancy of its *direction* was used ab initio to reduce from eight to six the number of equations that must be integrated. The integral corresponding to H's constant *magnitude* is

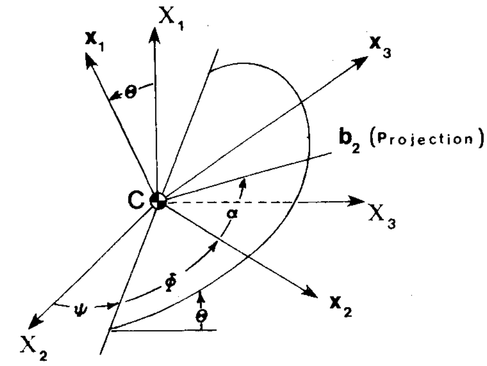

Fig. 2 Eulerian angles.

$$H^2=H_1^2+H_2^2+H_3^2 \qquad \text{const} \tag{8}$$

which represents a family of spheres in angular momentum space. It is also obvious from Eq. (1d) that an integral is

$$P_\alpha=\text{const} \tag{9}$$

A third integral can be found as follows. By multiplying Eq. (1a) by H_1/A, Eq. (1b) by H_2/B, and Eq. (1c) by H_3/C and then adding the results, one finds that

$$\dot{H}_1H_1/A+\dot{H}_2/B+\dot{H}_3H_3/C=[(B-C)/(BC)]H_2H_3P_\alpha/A \tag{10}$$

Since $\dot{H}_1=[(B-C)/(BC)]H_2H_3$ and P_α is constant, Eq. (10) can be integrated to obtain

$$T=\tfrac{1}{2}[(H_1-P_\alpha)^2/A+H_2^2/B+H_3^2/C] \qquad \text{const} \tag{11}$$

Now, $T\neq\mathfrak{T}$, the rotational kinetic energy of the system. However,[4]

$$\mathfrak{T}=T+\tfrac{1}{2}P_\alpha^2/B_1 \tag{12}$$

Thus, Eq. (11) is an "energy" integral that could have been derived using the integral (9), and the fact that the system considered is conservative. For A, B, and C specified, Eq. (11) represents, in angular-momentum space, a two-parameter (T and P_α) family of ellipsoids with centers at (P_α, 0,0).

The integrals (8), (9), and (11) can be used to geometrically describe part of the motion of the system, namely, the motion of the angular momentum vector as viewed by an observer in the $Cx_1x_2x_3$ system. As seen by such an observer, the terminus of the angular momentum vector describes the curve, or one of two possible curves, of intersection of the sphere and ellipsoid corresponding to a particular spacecraft configuration, and a given set of initial conditions. The interested reader may consult Ref. 7 for examples of such curves.

Other geometrical aspects of the motion are the variations of H_2 and H_3 with H_1. From Eqs. (8) and (11), one may obtain the relations,

$$H_2^2(H_1)=\frac{B(A-C)}{A(B-C)}\left[\left(H_{10}+\frac{C}{A-C}P_\alpha\right)^2 + \frac{A(B-C)}{B(A-C)}H_{20}^2-\left(H_1+\frac{C}{A-C}P_\alpha\right)^2\right] \tag{13a}$$

and

$$H_3^2(H_1)=\frac{C(A-B)}{A(B-C)}\left[\left(H_1+\frac{B}{A-B}P_\alpha^2\right)\right.$$
$$\left.+\frac{A(B-C)}{C(A-B)}H_{30}^2-\left(H_{10}+\frac{B}{A-B}P_\alpha\right)^2\right] \quad (13b)$$

where H_{j0} is the value of H_j at some time t_0. Equations (13) define families of parabolas. For $C>B>A$ and $P_\alpha \geq 0$, Eq. (13a) defines parabolas that are "concave down" (see Fig. 3) and have maxima at $H_1=-[C/(A-C)]P_\alpha$. In this case, the other parabolas [Eq. (13b)] are "concave up" with minima at $H_1=-[B/(A-B)]P_\alpha$. Unless $P_\alpha=0$ (or $B=C$), corresponding maxima and minima do not occur at the same value of H_1. Motion is restricted to values of H_1 for which both H_2^2 and H_3^2 are non-negative.

Exact Analytical Solution

A complete analytical description of the attitude of the spacecraft consists of expressions for the state variables $H_1,H_2,H_3,P_\alpha,\Psi,\Theta,\Phi$, and α as functions of time. Due to the assumption of torque-free motion, expressions for Θ and Φ can be obtained from those for H_1,H_2, and H_3 [see Eqs. (4) and (5)]. However, it is also possible to obtain the solution for Φ another way. This will be considered shortly. Because $P_\alpha=$const, expressions for H_1,H_2,H_3,Ψ,Φ, and α are those which will be found.

Solution for H_1

From Eq. (1a) and Eqs. (13), one may obtain

$$\dot H_1=[(A-C)(A-B)/(ABC)]f_2(H_1)f_3(H_1) \quad (14)$$

where

$$f_2(H_1)=\left(H_{10}+\frac{C}{A-C}P_\alpha\right)^2$$
$$+\frac{A(B-C)}{B(A-C)}H_{20}-\left(H_1+\frac{C}{A-C}P_\alpha\right)^2 \quad (15a)$$

and

$$f_3(H_1)=\left(H_1+\frac{B}{A-B}P_\alpha\right)^2$$
$$+\frac{A(B-C)}{C(A-B)}H_{30}^2-\left(H_{10}+\frac{B}{A-B}P_\alpha\right)^2 \quad (15b)$$

The solution for H_1 depends upon the nature of the zeros of the functions f_2 and f_3; i.e., upon the zeros of H_2 and H_3. The product f_2f_3 is obviously a quartic in H_1. Five types of solutions (see Ref. 1, p. 226) are physically possible. These correspond to the following characteristics of the roots of the quartic equation, $f_2f_3=0$: 1) four different real roots, 2) two different real roots and a pair of complex roots, 3) two different real roots and one double real root, 4) one simple real root and a triple real root, and 5) two double real roots. The "double" and "triple" root cases are special cases of the first two. Hence, emphasis shall here be placed on the first two types. These will be referred to as case 1 and case 2, respectively. Of these, the first is possibly of the most interest because the second corresponds to "flat spin" of the spacecraft; i.e., rotation in which the angle Θ is near 90 deg, and H_3 is not zero during the motion because $f_3(H_1)$ has only complex zeros.

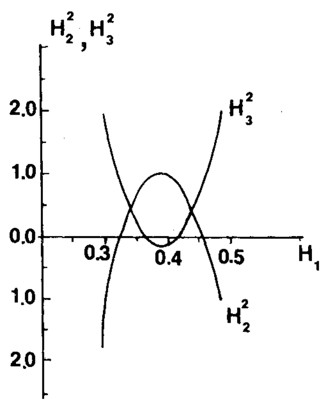

Fig. 3 Examples of $H_2^2(H_1)$ and $H_3^2(H_1)$.

Case 1: Four Real Roots ($a>b>c>d$) (see Ref. 8, pp. 107-124)

When both f_2 and f_3 have two distinct real zeros,

$$H_1=(D_1+D_2sn^2u)/(D_3+D_4sn^2u) \quad (16)$$

where the D_j, $j=1,2,3,4$, are constants which are defined in what follows, snu is a Jacobian elliptic function (sine amplitude u) of modulus,

$$k=[(a-b)(c-d)/(a-c)(b-d)]^{1/2} \quad (17)$$

and

$$u=\lambda(t-t_0) \quad (18)$$

in which

$$\lambda=(1/2A)[(A-B)(A-C)/(BC]^{1/2}[(a-c)(b-d)]^{1/2} \quad (19)$$

and t_0 is the initial time.

The D_j are determined by H_{10}, the initial value of H_1. Due to the nature of the curve(s) of intersection, H_{10} is never between b and c. Hence, there are two subcases. If $a\geq H_{10}\geq b$, $D_1=a(b-d)$, $D_2=d(a-b)$, $D_3=(b-d)$ and $D_4=(a-b)$. If $c\geq H_{10}\geq d$, $D_1=d(a-c)$, $D_2=a(c-d)$, $D_3=(a-c)$ and $D_4=(c-d)$.

Case 2: Two Real Roots and Two Complex Roots ($a>b$, $c=b_1+ia_1$ and $\bar c=b_1-ia_1$) (see Ref. 8, p. 133)

As mentioned above, if such a system of roots exists, the spacecraft is in a "flat-spin" state. The solution is of the form

$$H_1=(C_1+C_2cnu)/(C_3+C_4cnu) \quad (20)$$

where the C_j, $j=1,2,3,4$, are defined in the following, cnu is a Jacobian elliptic function (cosine amplitude u) of modulus,

$$k=\{[(a-b)^2-(A^*-B^*)^2]/(4A^*B^*)\}^{1/2} \quad (21)$$

and again $u=\lambda(t-t_0)$, but in this case,

$$\lambda=(1/A)[(A-C)(A-B)/(BC)]^{1/2}(A^*B^*)^{1/2} \quad (22)$$

The constants A^* and B^* are defined by the relations,

$$A^*=[(a-b_1)^2+a_1^2]^{1/2} \qquad B^*=[(b-b_1)^2+a_1^2]^{1/2} \quad (23)$$

where b_1 and a_1 are the real and imaginary parts, respectively, of the complex root c. The constants C_j, $j=1,2,3,4$, are defined as $C_1=aB^*+bA^*$, $C_2=bA^*-aB^*$, $C_3=A^*+B^*$ and $C_4=A^*-B^*$.

Solutions for H_2 and H_3

Explicit solutions for H_2 and H_3, which have not been given previously in the literature, may be obtained by using the solutions for H_1 given above in Eqs. (16) and (20).

Case 1: Four Real Roots

The solutions for H_2 and H_3 in this case may have several forms. If the zeros of H_2 are a and c,

$$H_2 = \pm\sigma_2 Q snu\ dnu/(D_3 + D_4 sn^2u) \tag{24a}$$

and

$$H_3 = \pm\sigma_3 R\ cnu/(D_3 + D_4 sn^2u) \tag{24b}$$

while, if the zeros of H_2 are a and d,

$$H_2 = \pm\sigma_2 Q snu/(D_3 + D_4 sn^2u) \tag{25a}$$

and

$$H_3 = \pm\sigma_3 R cnu\ dnu/(D_3 + D_4 sn^2u) \tag{25b}$$

Here,

$$dnu = [1 - k^2 sn^2 u]^{1/2}$$

$$\sigma_2 = B(A-C)/A(B-C) \qquad \sigma_3 = C(A-B)/A(B-C)$$

and Q and R will be defined subsequently. Additional possibilities exist, mathematically speaking; e.g., the case in which H_2 has zeros b and c. However, those given above are sufficient to show that H_2 and H_3 can be expressed explicitly in any particular case. If needed, the forms of the solution for H_2 and H_3 in the other cases can be found by interchanging H_2 and H_3 in Eqs. (24) and (25), carrying σ_2 and σ_3 along with H_2 and H_3.

When the zeros of H_2 are a and c (generally an oblate spacecraft) and $a \geq H_{10} \geq b$,

$$Q = [(a-c)(b-d)(a-d)(a-b)]^{1/2}$$
$$R = (b-d)[(a-b)(a-d)]^{1/2} \tag{26}$$

while, if $c \geq H_{10} \geq d$,

$$Q = [(a-d)(c-d)(b-d)(a-c)]^{1/2}$$
$$R = (a-c)[(a-d)(c-d)]^{1/2} \tag{27}$$

When the zeros of H_2 are a and d (generally a prolate spacecraft) and $a \geq H_{10} \geq b$,

$$Q = (a-d)[(b-d)(a-b)]^{1/2}$$
$$R = (b-d)[(a-c)(a-b)]^{1/2} \tag{28}$$

while, if $c \geq H_{10} \geq d$,

$$Q = (a-d)[(a-c)(c-d)]^{1/2}$$
$$R = (a-c)[(b-d)(c-d)]^{1/2} \tag{29}$$

Case 2: Two Real Roots and Two Complex Roots

When H_3 has complex zeros,

$$H_2 = \pm\sigma_2 Q snu/(C_3 + C_4 cnu) \tag{30a}$$

and

$$H_3 = \pm\sigma_3 R dnu/(C_3 + C_4 cnu) \tag{30b}$$

where

$$Q = (A^* B^*)^{1/2}(a-b) \qquad R = (C_4 C_1 - C_2 C_3)/(a-b) \tag{31}$$

Of course, the proper expressions for the modulus [Eq. (21)] and λ [Eq. (22)] must be used in Eqs. (30).

In the above expressions for H_2 and H_3, the sign (+ or −) on each of H_2 and H_3 must be chosen so that given initial conditions and the equations of motion are satisfied.

Solution for Ψ

The angle Ψ can be obtained by integrating Eq. (7). As long as $H_1 \neq H$, one can use the integrals (8) and (11) to write

$$\dot{\Psi} = \frac{H}{A} + \frac{2AT - (H+P_\alpha)^2}{2A}\,\frac{1}{H+H_1} + \frac{2AT - (H-P_\alpha)^2}{2A}\,\frac{1}{H-H_1} \tag{32}$$

Case 1: Four Real Roots

The solution for H_1 as given by Eq. (16) can be substituted into Eq. (32), and some algebraic reduction accomplished to obtain

$$\Psi = b_0(u-u_0) + b_1\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\alpha_1^2 sn^2u} + b_2\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\alpha_2^2 sn^2u} + \Psi_0 \tag{33}$$

where the b_j, $j = 0,1,2$, and the α_k, $k = 1,2$, are constants and $u_0 = \lambda t_0$. It is convenient in defining the b_j to let

$$c_1 = H/A \qquad c_2 = [2AT - (H+P_\alpha)^2]/(2A)$$
$$c_3 = [2AT - (H-P_\alpha)^2]/(2A) \qquad c_4 = D_4/(D_4H + D_2)$$
$$c_5 = (D_3D_2 - D_1D_4)/(D_3H + D_1)(D_4H + D_2) \tag{34}$$
$$c_6 = D_4/(D_4H - D_2)$$
$$c_7 = (D_3D_2 - D_4D_1)/(D_3H - D_1)(D_4H - D_2)$$

Then,

$$b_0 = (c_1 + c_2c_4 + c_3c_6)/\lambda \qquad b_1 = -c_3c_7/\lambda \qquad b_2 = c_2c_5/\lambda \tag{35a}$$

Also,

$$\alpha_1^2 = -(D_4H - D_2)/(D_3H - D_1)$$
$$\alpha_2^2 = -(D_4H + D_2)/(D_3H + D_1) \tag{35b}$$

The integrals in Eq. (33) are readily recognized as elliptic integrals of the third kind. Depending on the values of the parameters α_j, these integrals may be evaluated by referring to Ref. 8, pp. 232-237. Generally, $\Psi = \Lambda_\Psi (t - t_0)$ + periodic terms, where Λ_Ψ is the constant, mean, time-rate-of-change of Ψ. A part of Λ_Ψ comes from each of the elliptic integrals.

Case 2: Two Real Roots

When only two zeros of $\dot{H}_1^2$ are real, either H_2 or H_3 is never zero, but instead librates between maximum and minimum values. The only substantive changes in the solution for Ψ are that cnu appears in place of sn^2u and new constants analogous to the b_j, $j = 0,1,2$, must be defined. Explicitly, the form of the solution is

$$\Psi = d_0(u-u_0) + d_1\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\beta_1^2 cnu} + d_2\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\beta_2^2 cnu} + \Psi_0 \tag{36}$$

where d_j and β_k have the same forms as the b_j and α_k, respectively, but in those forms C_m replaces D_m. Also, Eqs. (20) and (22) are to be used for k and λ.

The integrals appearing in Eq. (36) are also elliptic integrals of the third kind. They can be evaluated analytically by referring to Ref. 8, p. 215 and pp. 232-237. As in the previous case, Ψ is composed of secular and periodic terms.

Solution for Φ

Although the solution for Φ can be expressed as the arctangent of a function of Jacobian elliptic functions [see Eqs. (25) and (30)], that form of solution does not provide directly the mean time-rate-of-change of Φ which is of foremost importance in determining the existence of internal resonance conditions. A new alternate form of the solution has been found. To some, the new form may be somewhat less appealing because it involves elliptic integrals of the third kind. However, it is not any more complex than the solution for Ψ.

From Poisson's kinematic equations, one has

$$\dot{\Phi}=(H_I-P_\alpha)/A-\dot{\Psi}\cos\Theta \tag{37}$$

and, by using the available integrals and Eq. (32), one can put Eq. (37) into the form,

$$\dot{\Phi}=P_\alpha/A+\frac{2AT-(H+P_\alpha)^2}{2A}\,\frac{1}{H+H_I}-\frac{2AT-(H-P_\alpha)^2}{2A}\,\frac{1}{H-H_I} \tag{38}$$

The solution for Φ in each of the two cases considered above can be obtained by integrating Eq. (38).

Case 1: Four Real Roots

The solution for Φ has the same general form as that for Ψ. Explicitly,

$$\Phi=p_0(u-u_0)+p_1\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\alpha_1^2 sn^2u}+p_2\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\alpha_2^2 sn^2u}+\Phi_0 \tag{39}$$

where

$$p_0=[(P_\alpha/A)-c_3c_6+c_2c_4]/\lambda$$

$$p_1=c_3c_7/\lambda=-b_1 \qquad p_2=c_2c_5/\lambda=b_2 \tag{40}$$

Case 2: Two Real Roots

The statements made above regarding the solution for Ψ hold for the Φ solution also.

In either of the two cases, the solution for Φ is of the form $\Phi=\Lambda_\Phi(t-t_0)$ + periodic terms where Λ_Φ is the counterpart of Λ_Ψ.

Solution for α

The solution for α is more easily obtained since $\dot{\alpha}$ is already explicitly in terms of H_I.

Case 1: Four Real Roots

In this case,

$$\alpha=a_0(u-u_0)+a_1\int_{u_0}^{u}\frac{\mathrm{d}u}{1-\alpha_3^2 sn^2u}+\alpha_0 \tag{41}$$

where

$$a_0=P_\alpha[(A+B)/B_I-D_2/D_4]/(A\lambda) \tag{42a}$$

$$a_1=(D_2D_3-D_1D_4)/(AD_3D_4\lambda) \tag{42b}$$

and

$$\alpha_3^2=-D_4/D_3 \tag{43}$$

Case 2: Two Real Roots

The statements made above regarding the differences in the two-real-root solutions and four-real-root solutions for Ψ and Φ are also applicable to the solutions for α.

As with Ψ and Φ, α may, in general, be expressed as $\alpha=\Lambda_\alpha(t-t_0)$ + periodic terms, where Λ_α is the counterpart of Λ_Ψ and Λ_Φ.

Internal Resonances

A practical application of the solutions for the angles Φ and α is that of determining conditions for the existence of internal resonances involving these angles. As shown in Ref. 4, the excitation terms due to dynamic imbalance of the otherwise axisymmetric body contain trigonometric functions with argument $\Phi-\alpha$. When the mean time-rates-of-change of Φ and α are equal, a nonlinear, internal resonance condition exists.

An approximate internal resonance curve (initial value of ω_1 vs initial value of $-\Omega$), defined by only the inertia characteristics of the system, has been given in Ref. 4. *Exact* resonance curves can be obtained by determining, for a given set of inertia characteristics, and a given angular momentum magnitude, the initial values of ω_1 and Ω which make $\Lambda_\Phi=\Lambda_\alpha$. This must be done in an iterative manner.

The values $A=407$ kg-m², $B=373$ kg-m², $C=339$ kg-m² and $B_I=68$ kg-m², which are those used in Refs. 3 and 4, along with $H=2400$ kg-m²/s, have been used to construct the exact resonance curves shown in Fig. 4. Since H is constant, each point on a curve represents a different energy state. The exact curves were obtained by fixing Ω_0 and varying ω_{10} until $\Lambda_\Phi=\Lambda_\alpha$ (numerically to three significant figures). For each pair (ω_{10},Ω_0) infinitely many resonance curves may be found, for there is only one constraint connecting the four variables ω_1, ω_2, ω_3, and Ω; i.e., H=const. Those shown are for ω_{20} and $\omega_{30}=0$, the extreme cases. These curves define a "region of resonance."

The approximate "curve" is a straight line, but the exact curves are not. The $\omega_{20}=0$ curve is slightly concave upward, while the $\omega_{30}=0$ curve is more concave. The two exact curves originate very near the bounding line $H_I=H$. The smallest nutation angle at which resonance was found to exist is 3.58 deg. Points further from the starting point along either exact curve correspond to larger initial nutation. For the same kinetic energy, a point on each of the curves can be identified. Furthermore, the equal energy points can be connected by a curve that passes through the resonance region.

For $\omega_{20}=0$ and certain initial values of Ω, there are multiple values of ω_{10} that correspond to resonance. This implies that if energy is added while ω_2 is near zero, a "jump" in ω_1 to a lower value may occur with little change in Ω. However, while ω_3 is near zero, no such "jump" is possible. This is in agreement with the discovery by Scher and Farrenkopf[3] that by adding energy when the nutation angle is smallest (ω_3 near zero), and allowing friction to expend energy when nutation angle is largest (ω_2 near zero), the resonance region can be transversed successfully using a low-torque despin motor.

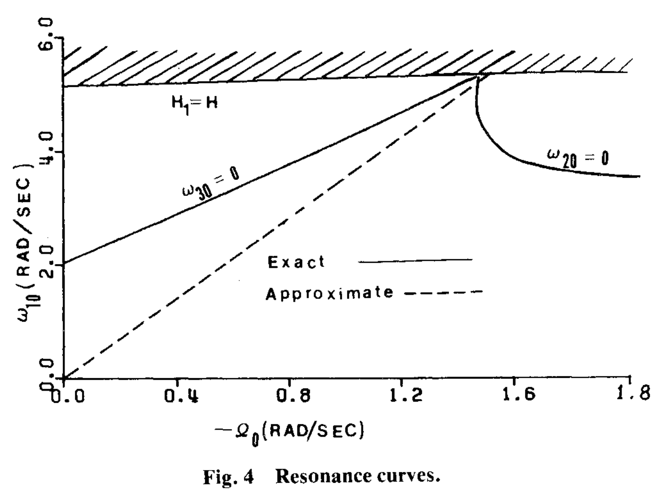

Fig. 4 Resonance curves.

Conclusion

An exact, closed-form, analytical solution for the attitude motion of an asymmetric gyrostat has been obtained. The solution is "complete" in the sense that all the Eulerian angles, and the angle of relative rotation, have been expressed in terms of Jacobian elliptic functions and elliptic integrals for the two cases of principal interest. The quadratures for Ψ, Φ, and α can also be evaluated analytically in special cases involving equal roots.

The fact that Φ can be expressed as either the arctangent of a function of Jacobian elliptic functions, or a linear term plus two elliptic integrals of the third kind is mathematically very interesting. It implies that there exist cases in which elliptic integrals of the third kind can be expressed in terms of elliptic functions. The elliptic integral form of the solution is more useful if the mean time-rate-of-change of Φ is needed.

As an application, the solutions for Φ and α were used to construct exact, nonlinear, internal, resonance curves, which define a "region of resonance" for a typical asymmetric dual-spin spacecraft. The delineation of such regions is an important part of the dual-spin spacecraft design process.

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